Linear Transformations

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Overview

Let A be an $m \times n$ matrix and x be an n-dimensional vector.

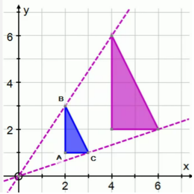
When A multiplies x, we can think of it as **transforming** that vector into a new vector Ax. This happens at every point x of the n-dimensional space $\mathbb{R}^n$.

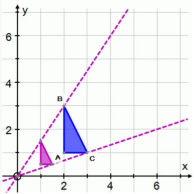
The whole space is transformed, or "mapped into," by the matrix A.

We disuss transformtion of this kind, in details.

Streching \ Positive Scaling

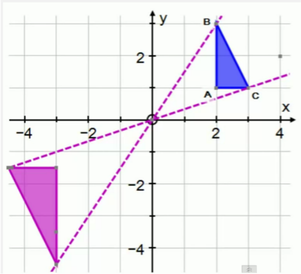
We start with four examples of the transformations that come from matrices. A multiple of the identity matrix, $A = cI$, **streches** every vector by the same factor c. The whole space expands or contracts (or somehow goes through the origin and out the opposite side, when c is negative).

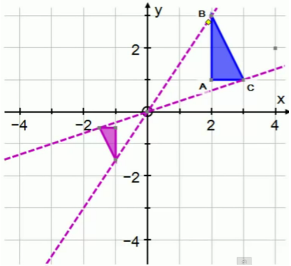




Positive Scaling

Enlargement / Shrink Scalar Factor $k > 0$, Centre $(0, 0)$





Negative Scaling

Enlargement / Shrink Scalar Factor $k < 0$, Centre $(0, 0)$

Rotation

A **rotation** matrix turns the whole space around the origin. The following example turns all vectors in the triangle with vertices $A(2,1)$, $B(2,3)$ and $C(3,1)$ through 90°.

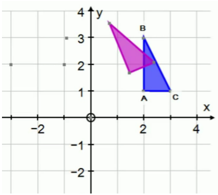


Figure 1

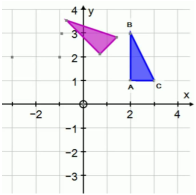


Figure 2

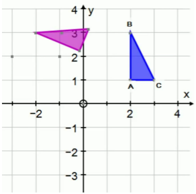


Figure 3

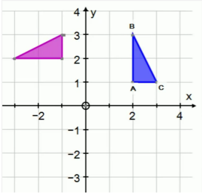
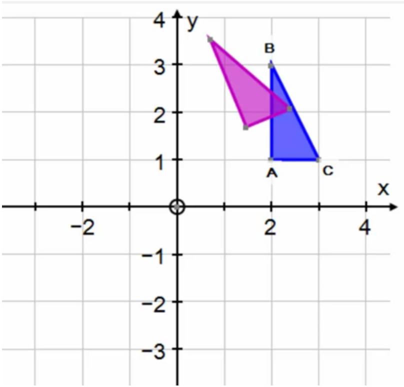
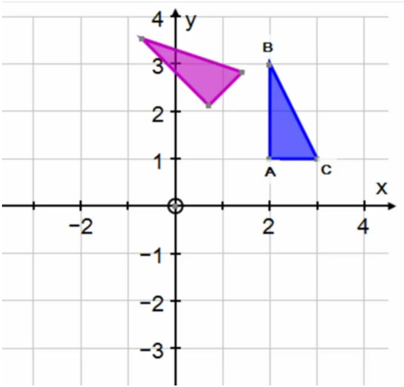


Figure 4

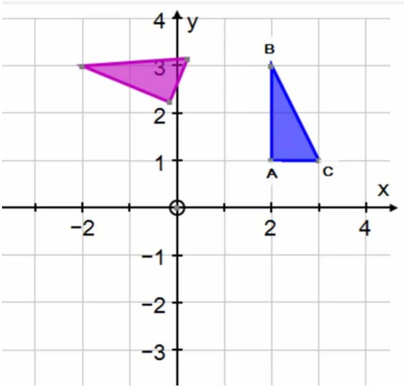
Rotation by 270° : Figure 1



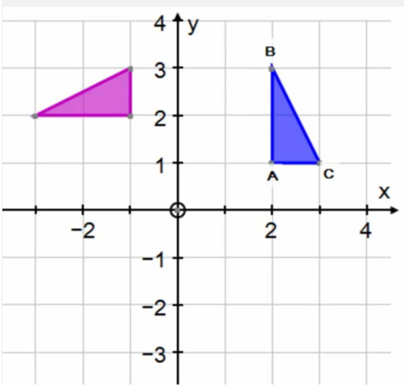
Rotation by 270° : Figure 2



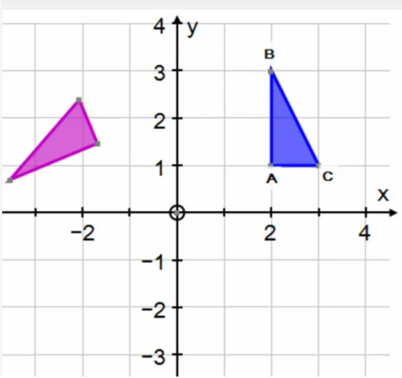
Rotation by 270° : Figure 3



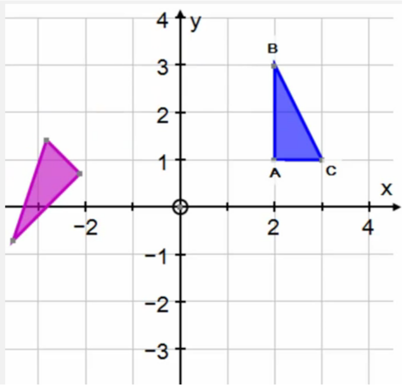
Rotation by 270° : Figure 4



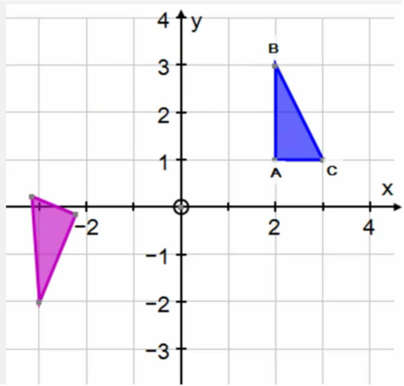
Rotation by 270° : Figure 5



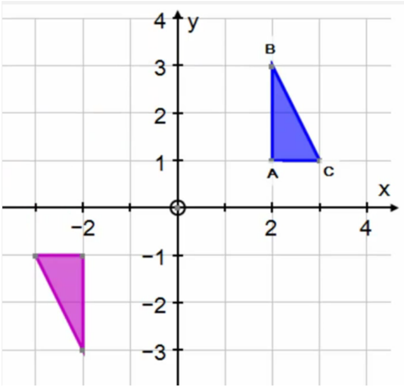
Rotation by 270° : Figure 6



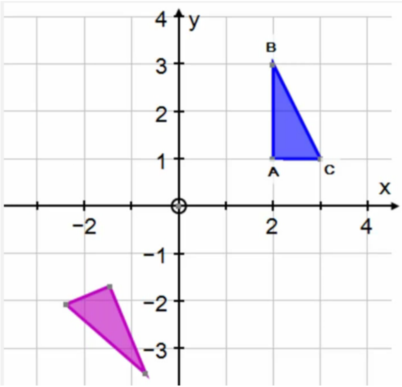
Rotation by 270° : Figure 7



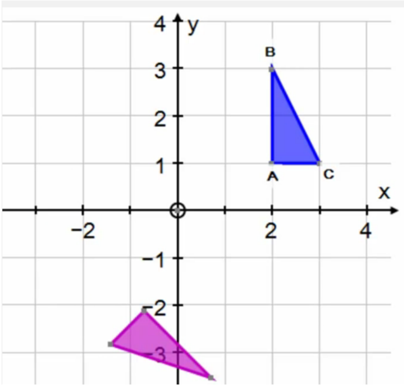
Rotation by 270° : Figure 8



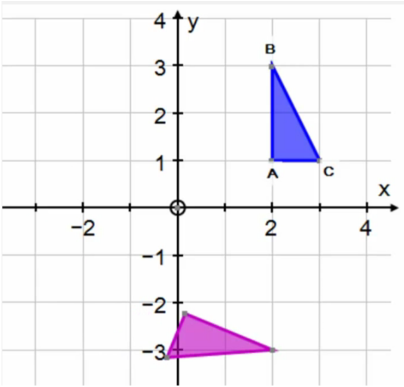
Rotation by 270° : Figure 9



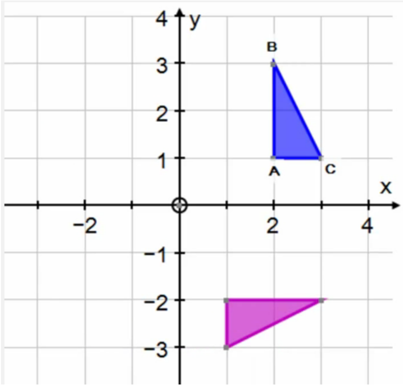
Rotation by 270° : Figure 10



Rotation by 270° : Figure 11

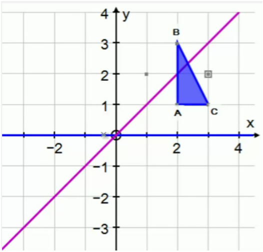


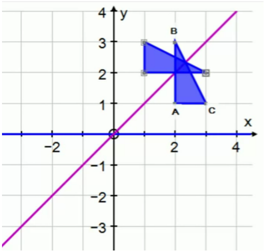
Rotation by 270° : Figure 12



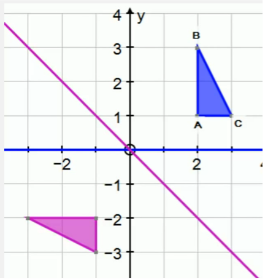
Reflection about the line $y = x$

A **reflection** matrix transforms every vector into its image on the opposite side of a mirror. In this example the mirror is the 45° line $y = x$, and a point $(2, 2)$ is unchanged. A point like $(2, 1)$ is reversed to $(1, 2)$. On a combination like $(1, 1) + (2, 1) = (3, 2)$, the matrix leaves one part and reverses the other part. The reflection matrix is also a permutation matrix! It is algebraically so simple, sending (x, y) to (y, x).





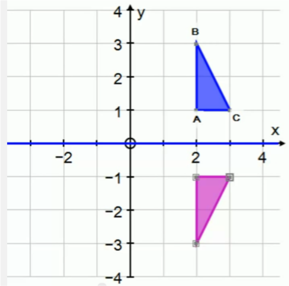
Reflection about the line $y = -x$



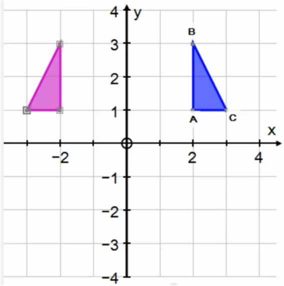
Reflection

About the line $y = -x$

Reflections About x and y axes



Reflection
About x-axis



Reflection
About y-axis

Reference

- Gilbert Strang, "*Linear Algebra and its Applications*", Cengage Learning, New Delhi, 2006.