Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal

sam.nitk.ac.in nitksam@gmail.com

Advanced Linear Algebra (MA 409)
Problem Sheet - 2

Vector Subspaces

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1. Label the following statements as true or false.
   (a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.
   (b) The empty set is a subspace of every vector space.

2. Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answer.
   (a) $W_1 = \Big\{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\Big\}$
   (b) $W_2 = \Big\{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\Big\}$
   (c) $W_3 = \Big\{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\Big\}$
   (d) $W_4 = \Big\{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\Big\}$
   (e) $W_5 = \Big\{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\Big\}$
   (f) $W_6 = \Big\{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1a_2a_3 = 0\Big\}$

3. Is the set $W_1 = \Big\{(a_1, a_2, \dots, a_n) \in F^n : a_1 + a_2 + \cdots + a_n = 0\Big\}$ a subspace of F^n?

4. Is the set $W_2 = \Big\{(a_1, a_2, \dots, a_n) \in F^n : a_1 + a_2 + \cdots + a_n = 1\Big\}$ a subspace of F^n?

5. Is the set $W = \Big\{f(x) \in P(F) : f(x) = 0 \text{ or } f(x) \text{ has degree n}\Big\}$ a subspace of $P(F)$ if $n \geq 1$?

6. Consider the vector space $P(\mathbb{R})$ over the field $\mathbb{R}$. Which of the following subsets are subspaces of $P(\mathbb{R})$?
   (a) the set of all polynomials of degree n ;
   (b) the set of all polynomials of degree less than or equal to n ;
   (c) the set of all polynomials of degree greater than or equal to n ;
   (d) $\Big\{p(x) \in P(\mathbb{R}) : p(0) = 2017\Big\}$;
   (e) $\Big\{p(x) \in P(\mathbb{R}) : p(0) = 0\Big\}$;
   (f) $\Big\{p(x) \in P(\mathbb{R}) : p(1729) = p(1887)\Big\}$.

7. Let S be a nonempty set and F be a field. Prove that for any $s_0 \in S$, $\Big\{f \in \mathcal{F}(S, F) : f(s_0) = 0\Big\}$ is a subspace of $\mathcal{F}(S, F)$.

8. Let S be a nonempty set and F be a field. Let $\mathcal{C}(S,F)$ denote the set of all functions $f \in \mathcal{F}(S,F)$ such that $f(s) = 0$ for all but a finite number of elements of S. Prove that $\mathcal{C}(S,F)$ is a subspace of $\mathcal{F}(S,F)$.

9. Is the set of all differentiable real-valued functions defined on $\mathbb{R}$ a subspace of $C(\mathbb{R})$? Justify your answer.

10. Let $C^n(\mathbb{R})$ denote the set of all real-valued functions defined on the real line that have a continuous n^{th} derivative. Prove that $C^n(\mathbb{R})$ is a subspace of $F(\mathbb{R},\mathbb{R})$.

11. Let V be a vector space and W a subset of V. The following are equivalent :

    (a) W is a subspace of V;

    (b) $0 \in W$, and whenever $a \in F$ and $x, y \in W$, then $ax \in W$ and $x + y \in W$;

    (c) $W \neq \emptyset$, and, whenever $a \in F$ and $x, y \in W$, then $ax \in W$ and $x + y \in W$;

    (d) $0 \in W$ and $ax + y \in W$ whenever $a \in F$ and $x, y \in W$.

12. Let F_1 and F_2 be fields. A function $g \in \mathcal{F}(F_1, F_2)$ is called an **even function** if $g(-t) = g(t)$ for each $t \in F_1$ and is called an **odd function** if $g(-t) = -g(t)$ for each $t \in F_1$. Prove that the set V_e of all even functions in $\mathcal{F}(F_1, F_2)$ and the set V_o of all odd functions in $\mathcal{F}(F_1, F_2)$ are subspaces of $\mathcal{F}(F_1, F_2)$. Also prove that $V_e + V_o = \mathcal{F}(F_1, F_2)$ and $V_e \cup V_o = \{0\}$.

    Consider the vector space $\mathcal{F}(\mathbb{C},\mathbb{C})$ over the field $\mathbb{C}$. Which of the following subsets are subspaces of $\mathcal{F}(\mathbb{C},\mathbb{C})$?

    (a) the set of all functions f such that $f(0) = 0$;

    (b) the set of all real valued functions ;

    (c) the set of all continuous functions.

13. Let W_1 and W_2 be subspaces of a vector space V.

    (a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

    (b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

14. Show that F^n is the direct sum of the subspaces

    $$W_1 = \Big\{(a_1, a_2, \ldots, a_n) \in F^n : a_n = 0\Big\}$$

    and

    $$W_2 = \Big\{(a_1, a_2, \ldots, a_n) \in F^n : a_1 = a_2 = \cdots = a_{n-1} = 0\Big\}.$$

15. Let W_1 denote the set of all polynomials $f(x)$ in $P(F)$ such that in the representation

    $$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

    we have $a_i = 0$ whenever i is even. Likewise let W_2 denote the set of all polynomials $g(x)$ in $P(F)$ such that in the representation

    $$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

    we have $b_i = 0$ whenever i is odd. Prove that $P(F) = W_1 \oplus W_2$.

16. Consider the vector space $M_{n\times n}(\mathbb{R})$ over the field R. Which of the following subsets are subspaces of $M_{n\times n}(\mathbb{R})$?

    (a) the set of all matrices whose entries are non-negative ;

    (b) the set of all invertible matrices ;

    (c) the set of all symmetric matrices ;

(d) the set of all skew-symmetric matrices ;

(e) the set of all upper triangular matrices ;

(f) the set of all matrices with trace zero.

17. In $M_{m\times n}(F)$ define $W_1 = \Big\{A \in M_{m\times n}(F) : A_{ij} = 0$ whenever $i > j\Big\}$ and $W_2 = \Big\{A \in M_{m\times n}(F) : A_{ij} = 0$ whenever $i \leq j\Big\}$. Show that $M_{m\times n}(F) = W_1 \oplus W_2$.

18. Let V denote the vector space consisting of all upper triangular $n \times n$ matrices, and let W_1 denote the subspace of V consisting of all diagonal matrices. Show that $V = W_1 \oplus W_2$, where $W_2 = \Big\{A \in V : A_{ij} = 0$ whenever $i \geq j\Big\}$.

19. A matrix M is called **skew-symmetric** if $M^t = -M$. Clearly, a skew-symmetric matrix is square. Let F be a field. Prove that the set W_1 of all skew-symmetric $n \times n$ matrices with entries from F is a subspace of $M_{n\times n}(F)$. Now assume that F is not of characteristic 2, and let W_2 be the subspace of $M_{n\times n}(F)$ consisting of all symmetric $n \times n$ matrices. Prove that $M_{n\times n}(F) = W_1 \oplus W_2$.

20. Let F be a field that is not of characteristic 2. Define

$$W_1 = \Big\{A \in M_{n\times n}(F) : A_{ij} = 0 \text{ whenever } i \leq j\Big\}$$

and W_2 to be the set of all symmetric $n \times n$ matrices with entries from F. Both W_1 and W_2 are subspaces of $M_{n\times n}(F)$. Prove that $M_{n\times n}(F) = W_1 \oplus W_2$.

21. Is the set $W_3 = \Big\{(a_1, a_2, a_3) : 2a_1 - 3a_2 + \sqrt{2}a_3 = 0, a_1 - 5a_3 = 0\Big\}$ a subspace of F^3?

22. Show that the following subsets of $\mathbb{R}$ form subspaces of $\mathbb{R}$ over $\mathbb{Q}$:

(i) $\mathbb{Q}$

(ii) $\Big\{\alpha + \beta\sqrt{2} + \gamma\sqrt{3} : \alpha, \beta, \gamma \in \mathbb{Q}\Big\}$.

23. In each of the following, find out whether the subsets given form subspaces of the vector space V.

(a) $V = \mathbb{R}^2$, $W_1 =$ the set of all (x_1, x_2) such that $x_1 \geq 0$ and $x_2 \geq 0$ and $W_2 =$ the set of all (x_1, x_2) such that $x_1 x_2 \geq 0$.

(b) $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$, $W_1 = \Big\{f : f \text{is monotone}\Big\}$, $W_2 = \Big\{f : f(2) = (f(5))^2\Big\}$ and $W_3 = \Big\{f : f(2) = f(5)\Big\}$. Note that monotone means either non-decreasing or non-increasing.

(c) $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$, $V =$ the set of all those functions whose range is finite (i.e., the function takes finitely many values).

(d) $V = \mathcal{F}(X, \mathbb{R})$, where X is the set of all positive integers and W=the set of all f such that the sequence $\{f(1), f(2), \ldots\}$ converges.

(e) $V = P_5(F), W = \Big\{p \in V : p = 0$ or degree $p \geq 2\Big\}$.

(f) $V = P(\mathbb{R})$ and $W = \Big\{p \in V : p(5) = 0\Big\}$.

(g) $V = P(\mathbb{R})$ and $W = \Big\{p \in V : p(5) \neq 2\Big\}$.

(h) $V =$ the power set of $\mathbb{R}$, $W =$ the set of all finite subsets of $\mathbb{R}$.

(i) $V = \mathbb{C}^n$ over $\mathbb{R}$, $W = \Big\{(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n : a_1$ is real $\Big\}$.

(j) $V = \mathbb{F}^n$, $W = \Big\{(a_1, a_2, \ldots, a_n) \in \mathbb{F}^n : a_2$ is rational $\Big\}$.

24. Let W be the set of all $(a_1, a_2, a_3, a_4, a_5)$ in $\mathbb{R}^5$ which satisfy

$$2a_1 - a_2 + \frac{3}{5}a_3 - a_4 = 0$$
$$a_1 + \frac{4}{3}a_3 - a_5 = 0$$
$$9a_1 - 3a_2 + 6a_3 - 3a_4 - 3a_5 = 0.$$

Find a finite set of vectors which spans W.

25. Let F be a field and let n be a positive integer $(n \geq 2)$. Let V be the vector space of all $n \times n$ matrices over F.

Which of the following sets of matrices A in V are subspaces of V ?

(a) all invertible A ;

(b) all non-invertible A ;

(c) all A such that $AB = BA$, where B is some fixed matrix in V ;

(d) all A such that $A^2 = A$.

26. Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector x in V there are unique vectors x_1 in W_1 and x_2 in W_2 such that $x = x_1 + x_2$.

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