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Theory of Complex Variables - MA 209
Problem Sheet - 5
Differentiable, Analytic and Harmonic Functions

1. Use definition to find the derivative for the given functions.
   (a) $f(z) = 9iz + 2 - 3i$
   (b) $f(z) = z - \frac{1}{z}$
   (c) $f(z) = z^4 - z^2$

2. The function $f(z) = |z|^2$ is continuous at the origin.
   (a) Show that f is differentiable at the origin.
   (b) Show that f is not differentiable at any point $z \neq 0$

3. Show that the given function is nowhere differentiable.
   (a) $f(z) = \bar{z}$
   (b) $f(z) = |z|$

4. Use L'Hopital's rule to compute the limit
   $\lim_{z \to \sqrt{2}i} z \frac{z^3+5z^2+2z+10}{z^5+2z^3}$

5. Determine the points at which the given function is not analytic.
   $f(z) = (\frac{(4+2i)z}{(2-i)z^2+9i})^3$

6. Let $f(z) = z^2$. Write down the real and imaginary parts of f and f′. Repeat the same for $f(z) = 3iz + 2$. Make a conjecture about the relationship between real and imaginary parts of f versus f′.

7. Show that the given function is not analytic at any point.
   $f(z) = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$

8. Show that the given function is analytic in an appropriate domain and find the derivative of the function in that domain.
   $(a) f(z) = e^{-x} cosy - ie^{-x} siny$
   $(b) f(z) = \frac{x^3+xy^2+x}{x^2+y^2} + i\frac{x^2y+y^3-y}{x^2+y^2}$
   $(c) f(z) = \frac{cos\theta}{r} - i\frac{sin\theta}{r}$

9. Show that the given function is not analytic at any point but is differentiable along the indicated curves and find the derivative of the function on the curve.
   $f(z) = x^3 - x + y + i(y^3 + 3x^2y - y)$:coordinate axes

10. Show that $f(z) = z^{\frac{1}{2}}$ is analytic on the domain $-\pi < \theta < \pi$.

11. Suppose f is analytic. Can $g(z) = \overline{f(z)}$ be analytic? Discuss and defend your answer with sound mathematics.

12. Verify that the given function u is harmonic in an appropriate domain D. Find its harmonic conjugate v and find an analytic function $f = u + iv$ satisfying the indicated conditions.
    $u(x, y) = xy + x + 2y - 5, f(2i) = -1 + 5i$

13. Show that $v(x, y) = \frac{x}{x^2+y^2}$ is harmonic in a domain D not containing the origin.

14. Verify that $u(x, y) = e^{x^2-y^2}$ is harmonic in an appropriate domain D.

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