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Problem Sheet 8

1. Think of a matrix A, as a linear map which takes the jth elements of the standard basis of $\mathbb{R}^n$ to the jth column C_j. How the column space is nothing other than $\texttt{Im}(A)$? Explain.

2. Prove or disprove: If A, B and C are pair-wise disjoint subsets of V such that $A \cup B$ and $A \cup C$ are bases of V, then $\texttt{Sp}(B) = \texttt{Sp}(C)$.

3. Let $x_1, x_2, \ldots, x_n$ be fixed distinct real numbers.

   (a) Show that $\ell_1(t), \ell_2(t), \ldots, \ell_n(t)$ form a basis of $\mathcal{P}_n$, where $\ell_i(t) = \Pi_{i \neq j}(t - x_j)$. This basis leads to what is known as *Lagrange's interpolation formula.* If $f(t) \in \mathcal{P}_n$ is written as $\sum_{i=1}^{n} \alpha_i \ell_i(t)$, show that $\alpha_i = f(x_i)/\ell_i(x_i)$.

   (b) Show that $\psi_1(t), \psi_2(t), \ldots, \psi_n(t)$ form a basis of $\mathcal{P}_n$, where $\psi_i(t) = 1$ and $\psi_i(t) = \Pi_{j=1}^{i=1}(t - x_j)$ for $i = 2, \ldots, n$. This basis leads to what is known as *Newton's divided difference formula.*

4. Extend $A = \{(1, 1, \ldots, 1)\}$ to a basis of $\mathbb{R}^n$.

5. Let S and T be subspaces of a vector space V with $d(S) = 2, d(T) = 3$ and $d(V) = 5$. Find the minimum and maximum possible values of $d(S + T)$ and show that every (integer) value between these can be attained.

6. Show that the distributive law

$$S \cup (T + W) = (S \cup T) + (S \cup W)$$

   is false for subspaces. However prove that it holds whenever $S \supseteq T$ or $S \supseteq W$. This latter result is known as the *modular law.*